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A Mathematical Model on the Two- phase Ocular Blood Flow in Arterioles with special reference to Eye disease Cataracts

Jayanti Tripathi[1], Archana Shukla[1] *, Virendra Upadhyay[2]

[1]Nehru Gram Bharti University (Deemed to Be University) Shashi Parisar Jhoothitali, Kotwa, Prayagraj - 221505, Uttar Pradesh, India

[2]Mahatma Gandhi Chitrakoot Gramodaya Vishwavidyalaya (MGCGV) Chitrakoot, Satna - 485334, Madhya Pradesh, India

*Corresponding Author: Archana Shukla (Email: archanashuklaald@gmail.com)

Abstract: *A two- phase model consisting of a core region of suspension of all the erythrocytes in plasma assumed to be a particle-fluid mixture and a layer of plasma has been proposed to represent blood flow in small diameter tubes. In this work we have formulated the Ocular blood flow in arterioles. As we know that the arterioles are remote from heart. The viscosity increases in the arterioles due to formation of Roulex along axis of red blood cells. V. upadhayay has assumed that the blood flow as two phased. He has also applied the Herschel Bulkley Non Newtonian model in Bio fluid mechanics. In this paper we have collected a clinical data in case of Ocular disease Cataracts. The graphical presentation for a particular parametric value is very close to clinical observation. The overall presentation is in Tensorial form. The roll of Hematocrit is explicit in determination of blood pressure in case of Ocular disease Cataracts.*

Keywords: *Hemoglobin, Blood pressure drop, Hematocrit*

INTRODUCTION

The percentage of volume covered by blood cells in whole blood is called Hematocrit and equal to three times of hemoglobin concentration. (1).Two phase of Ocular blood flow is a study of measuring the blood pressure if hemoglobin known.

There are a lot of work in this field but V. Upadhaya (2001) discussed a some phenomena in two phase blood flow gave an idea on the two phase coronary blood flow in arterioles with a cardiovascular disease Hypertension. V. Upadhaya gave an idea on whole circulatory system but in this work in this work we discuss two phase blood flow in arterioles on cardiovascular system by applying Herschel Bulkley Non Newtonian model.

Blood is a mixture of particular cupules suspended on a non Newtonian fluid. The particular corpulse are Red blood cells (RBC), White blood cells (WBC) and Platelets. The fluid is Plasma which itself a

complex mixture of proteins and other intergradient in an aqueous base. The whole blood consist 98 % RBC and remaining 2 % WBC and Platelets cells which is ignorable so one phase of blood is Plasma and second phase is RBC.

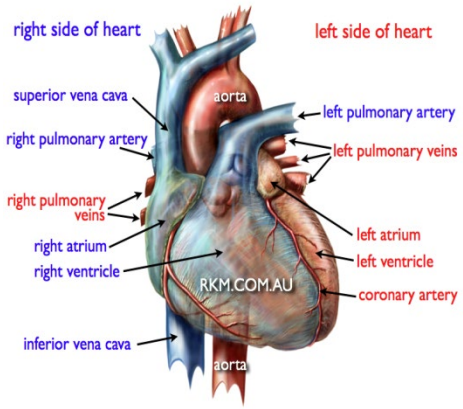

Fig. 1: *Anatomy of Heart*

The Heart is the highest oxygen consumption per tissue mass of all human organs. The coronary circulatory system starts from the left atrium, blood flows through the mitral valve in to the left ventricle. Contraction of the ventricle closes the mitral valve and opens the aortic valve at the entrance of the aorta. The first branches from the aorta occur just beyond the aortic valve still within the heart. Two opening lead to the right and left coronary arteries which supply blood to the network of capillaries that penetrate every portion of the heart.[(2)]

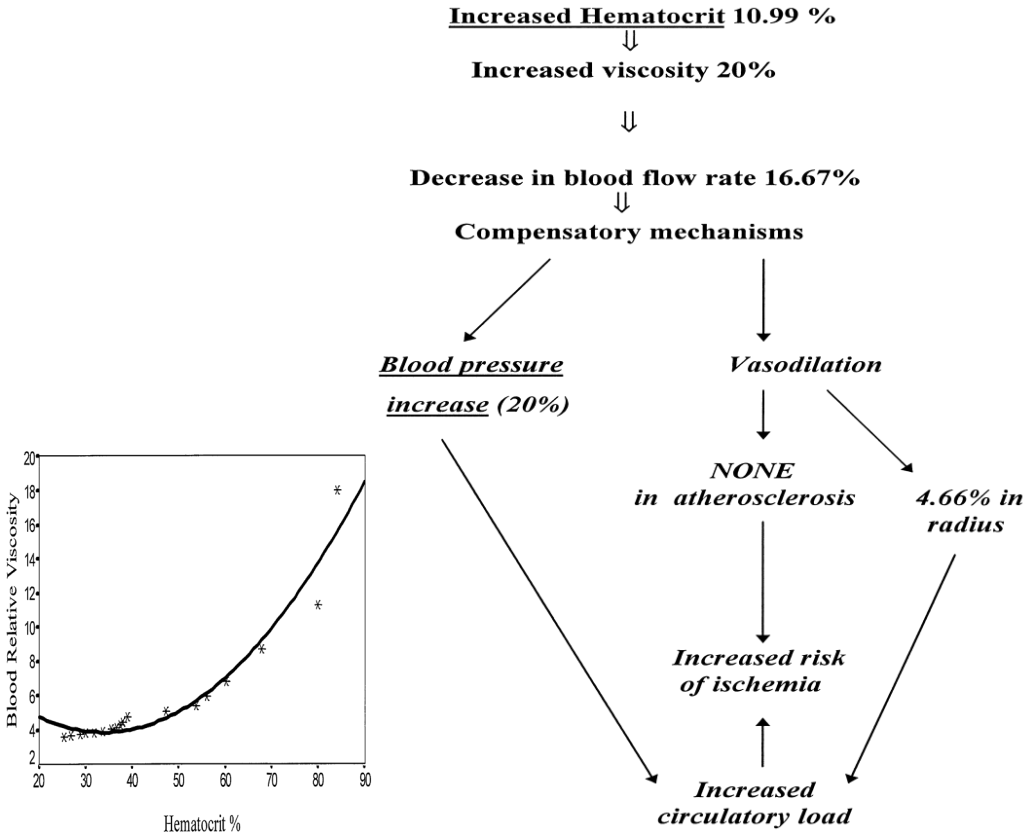

Fig. 2: *Hematocrit performances with Blood relative viscosity*

ANATOMY OF HUMAN EYE

The human eye is a complex sensory organ responsible for vision, featuring an approximate 2.3 cm diameter. It consists of three main layers (outer sclera/cornea, middle choroid/iris, and inner retina) and three chambers filled with fluid that focus light onto the retina to convert it into neural signals.

Table 1: *Components of human eye*

Component	Function
Cornea	Refracts light (focusing)
Iris	Controls light intensity (pupil size)
Lens	Accommodation (focusing)
Retina	Detects light/color, converts to signals
Optic Nerve	Transmits signals to brain
Sclera	Protects and maintains shape

A cataract is a common, age-related eye disease characterized by the clouding of the natural, normally clear lens, resulting in blurry, faded, or dull vision. It occurs when proteins in the lens break down and clumps together, often causing a gradual, painless decline in sight. Surgery is the only effective treatment. Common symptoms include blurry or foggy vision, poor night vision, sensitivity to light/glare, seeing halos around lights, and faded color perception. While primarily caused by aging (common after 40-50), they can be induced by diabetes, smoking, excessive alcohol consumption, UV light exposure, eye injuries, and long-term steroid use.

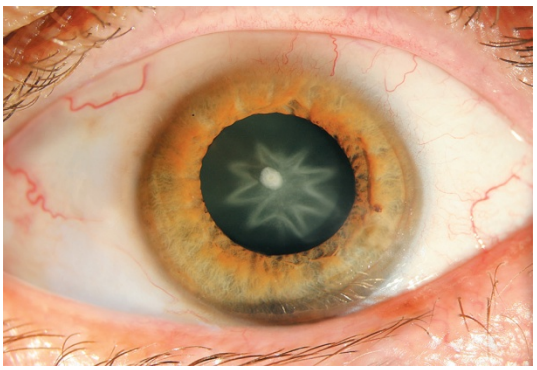

Fig. 3: *Human eye*

MATHEMATICAL MODELING

Bio-fluid equation for two-phase blood flow

Problem of blood flow in cardiovascular system is generally discuss in three dimensional orthogonal curvilinear co-ordinate system and briefly described as E^3 (Euclidean space).
According to Sherman I.W. and Sherman V.G. , blood is a mixture of plasma and blood cells enclosed

with a semi permeable membrane whose density is higher than plasma. These blood cells are uniformly distributed in plasma. Thus blood has been considered as a homogeneous mixture of two-phase.

Equation of Continuity for Two phase blood flow

The blood flow is affected by the presence of blood cells.[5] This effect is directly proportional to the volume occupied by blood cells. Consider the volume covered by blood cells in unit volume be X , then the volume covered by the plasma will be (1 - X). If the mass ratio of blood cells to plasma is r then

$$r = \frac{X\rho_c}{(1-X)\rho_p} \quad \dots (1)$$

Where ρ_c and ρ_p are densities of blood cells and plasma respectively. This mass ratio may be supposed to constant in present context. Campbell and Pitcher has presented a model in which these both phase blood cells and plasma move with the common velocity.(1958) Hence equation of continuity for two phase according to conservation of mass principle

$$\frac{\partial(X\rho_c)}{\partial t} + \left(X\rho_c V^i\right)_{,i} = 0 \quad \dots (2)$$

And

$$\frac{\partial(1-X)\rho_p}{\partial t} + \left[(1-X)\rho_p V^i\right], i = 0 \quad \dots (3)$$

Where, V is the common velocity of two phase blood cells and plasma.$(X\rho_c V^i)_{,i}$ is co-variant derivative of $(X\rho_c V^i)$ with respect to X^i and $\left[(1-X)\rho_p V^i\right], i$ is co-variant derivative of $\left[(1-X)\rho_p V^i\right]$ w.r.t. X^i.

We define the uniform density of the blood ρ_m as

$$\frac{1+r}{\rho_m} = \frac{r}{\rho_c} + \frac{1}{\rho_p} \quad \dots (4)$$

Then equation (2) and (3) combined as

$$\frac{\partial\rho_m}{\partial t} + \left(\rho_m V^i\right), i = 0 \quad \dots (5)$$

Equation of motion for two phase blood flow

The hydro dynamical pressure p between the two phase of blood can be assumed uniform because the both phases are always being equilibrium state in blood.[6] taking viscosity coefficient of blood cells to be $\eta_c$and applying the principle of conservation of momentum , we get the equation of motion for two phase blood cell as

$$X\rho_c \frac{\partial V^i}{\partial t} + \left(X\rho_c V^j\right), V^i_j = -Xp, {}_j g^{ij} + X\eta_c \left(g^{jk} V^i_k\right)_{,j} \quad \dots (6)$$

Similarly, taking the viscosity coefficient of plasma η_p , the equation of motion for plasma as

$$(1-X)\rho_p \frac{\partial V^i}{\partial t} + \left\{(1-X)\rho_p V^j\right\} V^i_j = -(1-X)p \ {}_j g^{ij} + (1-X)\eta_p (g^{jk} V^i_k)_{,j} \quad \dots (7)$$

Now adding equation (6) and (7) by using equation (4), the equation of motion for blood flow for both phases will be

$$\rho_m \frac{\partial V^i}{\partial t} + (\rho_m V^i), V_j^i = -p_{,j} + \eta_m (g^{jk} V_k^i)_{,j} \quad \ldots (8)$$

If the velocity of blood flow decreases, the viscosity increases. Arterioles, veinules and veins are relatively far enough from the heart. Due to this reason velocity of blood decreases. These vessels are very narrow so in this situation the blood cells line up on the axis to build up rouleaux. So here a very small yield stress is produced and the viscosity of blood is increased approximate ten times[7]

On two phase blood flow through veins, arterioles and veinules Herschel Bulkley law holds good and constitutive equation is as

$$T' = \eta_m e^{\eta} + T_p \qquad (T' \geq T_p)$$

And

$$e = 0 \qquad (T' < T_p)$$

Where, T_p is the yield stress. When strain rate $= 0$ $(T' < T_p)$, a core region is formed which flows just like a plug. Let the radius of the plug be r_p. The stress on the surface of plug will be T_p. Equating the forces acting on plug, we get

$$\rho \pi r_p^2 = T_p 2\pi r_p$$

$$r_p = 2\frac{T_p}{p} \quad \ldots (9)$$

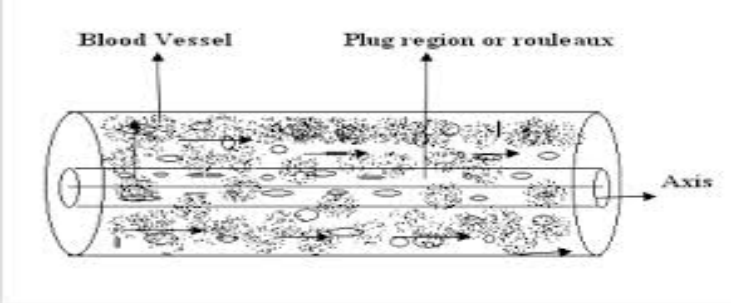

Fig. 4: *Herschel Bulkley Blood Flow*

The constitutive equation for rest part of the blood vessel is

$$T' = \eta_m e^n + T_p \text{ or } T' - T_p = \eta_m e^n = T_e$$

Where T_e is effective stress. Whose generalized form is

$$T^{ij} = -P g^{ij} + T_e^{ij}$$

where $T_e^{ij} = \eta_m (e^{ij})^n$ while $e^{ij} = g^{ij} V_k^i$

Now we explain the basic equation for Herschel Bulkley blood flow as :

Equation of Continuity –

$$\frac{1}{\sqrt{g\left(\sqrt{g V^i}\right)_i}} = 0$$

Equation of motion –

$$\rho_m \frac{\partial V_i}{\partial t} + \rho_m V^i V_j^i = -T_{e\,,j}^{\;ij} \quad \ldots (10)$$

All the symbols have their usual meanings.

RESULT AND DISCUSSION

The blood vessels are assumed as cylindrical hence all above governing equation have to transformed in to cylindrical co-ordinate

$$X^1 = r\,, X^2 = \theta\,, X^3 = Z$$

Therefore matrix of metric tensor is

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And matrix of conjugate metric tensor is

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Relation between contra variant and physical components of velocity of blood flow will be

$$\sqrt{g_{11}V^1} = V_r = V^1$$

$$\sqrt{g_{22}V^2} = V_\theta = rV^2$$

$$\sqrt{g_{33}V^3} = V_z = V^3$$

Now transform equation (9) and (10) in to cylindrical form and solve by power law model

$$\frac{dv}{dr} = \left(\frac{Pr}{2\eta_m}\right)^{1/n}$$

Where, P is pressure gradient i.e. $P = \frac{dp}{dz}$

Now replace the above equation for non plug region i.e. r to $r - r_p$

$$\frac{dv}{dr} = \left[\frac{P(r - r_p)}{2\eta_m}\right]^{1/n}$$

From equation (9) we get

$$\frac{dv}{dr} = \left(\frac{\frac{1}{2}Pr}{\eta_m}\right)^{1/n} \quad \ldots (11)$$

Substituting the value of T_p into equation (11)

$$\frac{dv}{dr} = \left(\frac{\frac{1}{2}Pr - \frac{1}{2}Pr_p}{\eta_m}\right)^{1/n}$$

$$\frac{dv}{dr} = -\left(\frac{P}{2\eta_m}\right)^{1/n} (r - r_p)^{1/n}$$

Integrating above equation with boundary condition V = 0 when r = R we get

$$V = -\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}} \frac{n}{n+1}\left[(R - r_p)^{\frac{1}{n}+1} - (r - r_p)^{\frac{1}{n}+1}\right] \ldots (12)$$

Above equation provide the formula for velocity for blood in arterioles,veinules and veins.

Putting $r = r_p$ to get the velocity V_p of plug

$$V_p = (P/2\eta_m)^{1/n} \frac{n}{n+1}\left[(R - r_p)^{\frac{1}{n}+1}\right] \ldots (13)$$

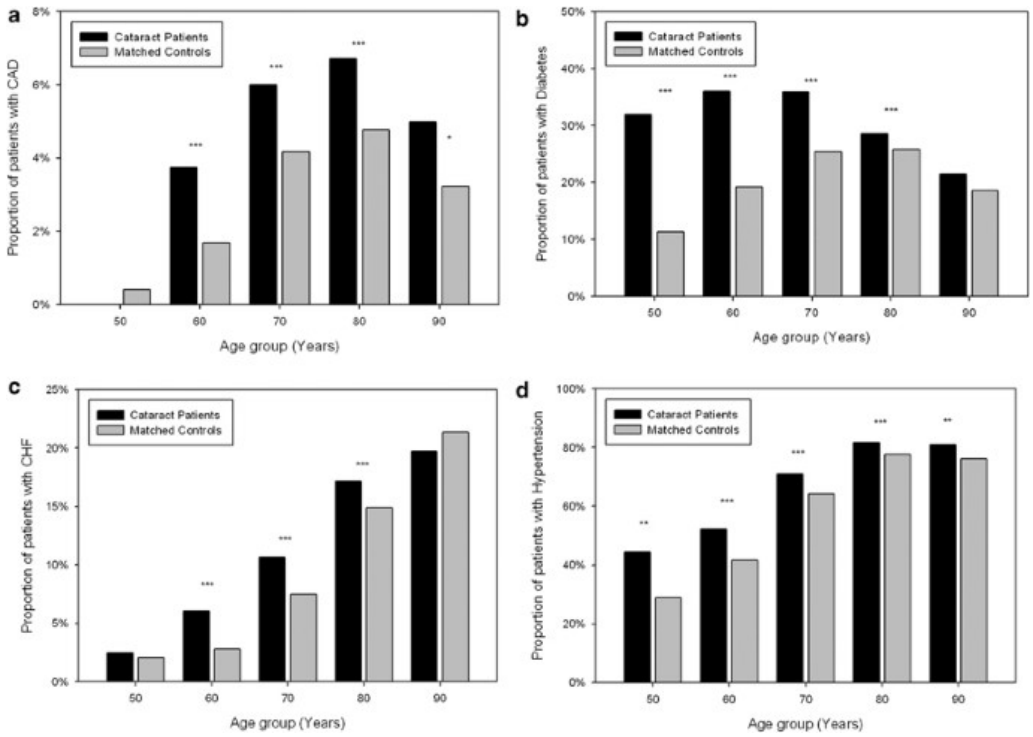

Fig. 5: *Different bar diagram with blood parameter versus age group.*

The clinical data is collected by Dr. A Ahmad, Vision Eye Care Hospital, India

Table 2: *Blood performance with various times*

Sr. No.	Date	Hemoglobin	Hematocrit	Blood Pressure
1	08 Aug 2025	8.01	24.05	140
2	17 Aug 2025	7.84	23.53	138
3	04 Aug 2025	7.61	22.83	136
4	05 Sep 2025	7.32	21.96	129

The blood flow flux in arterioles, venues and veins is

$$Q = \int_0^{r_p} 2\pi r V_p \, dr + \int_{r_p}^{R} 2\pi r V \, dr$$

By using (12) and (14) we get

$$Q = \frac{\pi n}{(n+1)}\left(\frac{P}{2\eta_m}\right)^{1/n} R^{\frac{1}{n}+3}\left[\frac{{r_p}^2}{R^2}\left(1-\frac{{r_p}^2}{R}\right)^{\frac{1}{n}+1} + \left(1+\frac{r_p}{R}\right)\left(1-\frac{r_p}{R}\right)^{\frac{1}{n}+2} - \frac{2\left(1-\frac{r_p}{R}\right)^{\frac{1}{n}+2}}{\frac{1}{n}+2} + \frac{2\left(1-\frac{r_p}{R}\right)^{\frac{1}{n}+3}}{\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)}\right]$$

Q = 1000 ml / min. R = 1 and r_p = 1/3

$\eta_p = 0.0015$ Pascal – sec. (according to Gustafson , Daniel R -1980)

And $\eta_m = 0.035$ Pascal – sec. (according to Glenn Elert -2010)

H = 15 and P = 90

$\eta_m = \eta_c X + \eta_p(1-X)$ where $X = \frac{H}{100}$

$$4299.36 = (857.1428)^{1/n}\left[\frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}\right]$$

By solving above equations we get n = 0.91843 and P =(5.7434) H + 3.8581

At H = 24.05 , P = 141.98

At H = 23.53 , P = 138.99

At H = 22.83 , P = 134.97

At H = 21.96 , P = 129.97

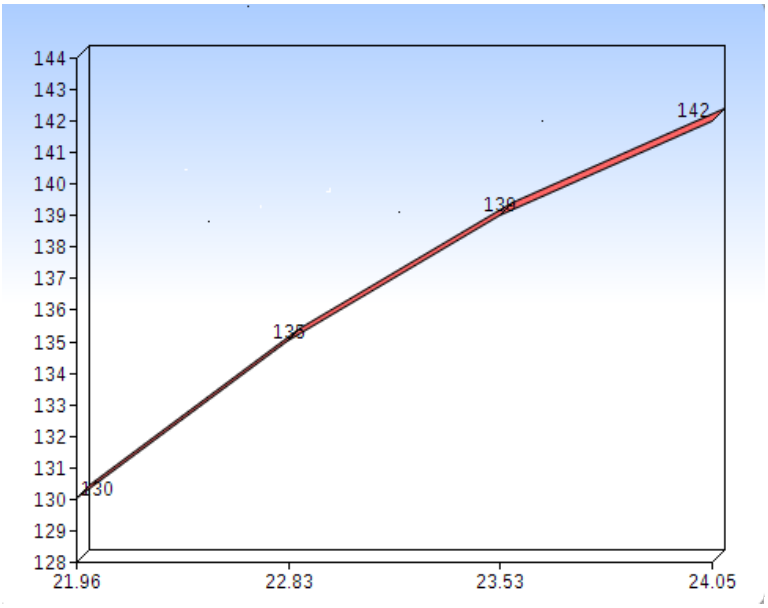

Fig. 6: *blood flow flux performance in arterioles*

CONCLUSION

The above result showing that when hematocrit increased then blood pressure also increase. The simple graph between blood pressure and hematocrit shows that the hematocrit is proportional to blood pressure in Ocular blood flow.A significant positive relationship exists between hematocrit and blood pressure, where elevated hematocrit levels increase blood viscosity and resistance, leading to higher blood pressure and increased risk of hypertension. Increased viscosity from higher red blood cell counts contributes to vascular strain, making it a potential predictive marker for hypertension.

- **Direct Correlation:** Hypertensive individuals often exhibit higher hematocrit levels compared to normotensive individuals.
- **Viscosity Effect:** High hematocrit increases blood viscosity, which causes increased vascular resistance and, consequently, elevated blood pressure.
- **Risk Factor:** High hematocrit is an independent risk factor for hypertension and cardiovascular issues,
- **Clinical Implications:** Elevated hematocrit acts as a potential prognostic marker for the early detection of hypertension.
- **Exceptions:** While high hematocrit correlates with high pressure, some studies suggest complex vascular responses where high viscosity might trigger vasodilatations in certain conditions.

REFERENCES

[1] Berkow, R. (1997). *The Merck manual of medical information*. Merck.

[2] Queen's Medical Centre. (n.d.). *Medical report*. Nottingham, NG7 2UH.

[3] Crillo, M. (1992). *Journal of Hypertension*, 20, 319–326.

[4] Yoldorom, C. E., & Onar. (1999). *American Journal of Hypertension*, 12, 739–743.

[5] Singh, P., & Upadhyay, K. S. (1986). A new approach for shock propagation in two-phase systems. *National Academy of Sciences*, 8(2).

[6] Ruch, T. C., & Patton, H. D. (1973). *Physiology and biophysics* (Vols. 2–3). W.B. Saunders.

[7] Guyton, A. C. (1981). *Medical physiology*. W.B. Saunders.

[8] Upadhayay, V. (2000). *Some phenomena in two-phase blood flow*.

[9] Mishra, R. S. (1990). *Tensors and Riemannian geometry*. Pothishala Pvt. Ltd., Allahabad.

[10] Kapur, J. N., & Gupta, R. C. (n.d.). Power-law fluid flow in the inlet length of a circular pipe.